

Gupta-Bleuler Photon Quantization in the SME

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Abstract

Photon quantization is implemented in the standard model extension (SME) using the Gupta-Bleuler method and BRST concepts. The quantization prescription applies to both the birefringent and non-birefringent CPT-even couplings. A curious incompatibility is found between the presence of the Lorentz-violating terms and the existence of a nontrivial conjugate momentum Π^0 yielding problems with covariant quantization procedure. Introduction of a mass regulator term can avoid the vanishing of Π^0 and allows for the implementation of a covariant quantization procedure. Field-theoretic calculations involving the SME photons can then be performed using the mass regulator, similar to the conventional procedure used in electrodynamics for infrared-divergence regulation.

INTRODUCTION

The Standard Model Extension (SME) is a framework for incorporating Lorentz and CPT-violating effects into the Standard Model [1] [2]. While issues of quantization in the fermion sector have received significant attention [3], the corresponding problem in the photon sector has not been sufficiently addressed due to a variety of difficulties that arise involving compatibility of gauge invariance, Lorentz violation, and non-orthogonality of the polarization vectors. Some recent progress has been made, involving a perturbative approach to the non-birefringent case [4], but the birefringent case remains largely unaddressed. A recent study on the subject involves a one-parameter model [5]. It is the goal of this paper to help fill in this gap in the literature and outline the procedure for quantizing the photon sector of the SME in the presence of CPT-even terms. The CPT-odd terms can be problematic even at tree level, so their quantization is not discussed in this paper.

Stringent bounds have been placed on Lorentz and CPT-violating parameters in the photon sector [6] with bounds on birefringent terms significantly stronger than non-birefringent terms, mainly due to the possibility of astrophysical tests for the former. Frequently birefringent terms are neglected in theoretical treatments due to this fact. In spite of this, the birefringent terms remain an interesting theoretical tool for studies of symmetry violation and present interesting challenges. Additional motivation arises from the fact that birefringence phenomena are in fact observed in various crystal substances.

Attempting to directly quantize the SME photon with zero mass runs into some serious impediments involving a lack of completeness of the polarization states, even when a gauge-fixing term is included in the lagrangian. One implication of this can be that the conjugate momentum Π^0 introduced by the gauge-fixing term can be driven identically to zero by the perturbed equations of motion. It turns out that by adding a photon mass, these issues can be alleviated and the covariant quantization procedure may be carried out consistently. For this reason, a photon mass is included in our calculations. Field-theoretic computations involving the photon should then be possible where the zero mass limit is taken at the end of any calculation, similar to what happens in the infrared divergence renormalization procedure in standard QED.

The paper is organized as follows. Section two presents the photon sector of the SME and analyzes properties of the solutions of the equation of motion. Section three gives

the quantization rules for the theory. A central result involving the modified orthogonality properties of the polarization vectors is used to invert the Fourier transform of the fields leading to a conventional algebra of raising and lowering operators that create and annihilate photon modes in the vacuum. Section four presents a computation of the propagator and includes a verification that the conventional Green's function method applies in this context. Section five discusses the Hamiltonian formalism and verifies the correct action of the energy and momentum operators on the state space. Section six presents a classification of k_F terms according to their duality properties. Finally, an explicitly solvable one-parameter example is presented to illustrate the formalism. The appendix includes an analysis of BRST symmetry to the theory involving an additional Stueckelberg field that can be used to find the physical subspace of the larger Hilbert space with indefinite metric.

PHOTON SECTOR OF THE SME

The photon sector of the SME consists of a CPT-even tensor $(k_F)^{\mu\nu\alpha\beta}$ and a CPT-odd vector $(k_{AF})^\mu$. Only the CPT-even tensor will be considered here as the CPT-odd tensor can lead to negative energy problems even in the tree-level theory [7]. Covariant gauge fixing in the photon sector can be implemented in the usual way leading to the lagrangian [8]

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}k_F^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} + \frac{1}{2}m^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \quad (1)$$

The choice $\xi = 1$ (Feynman gauge) is particularly convenient for the Gupta-Bleuler quantization procedure [9, 10] and is used in the rest of the paper. Using the Stueckelberg method, a mass term for the photon has also been included in (1) to avoid certain issues that can arise in the quantization of the massless theory, as we will see below. The equation of motion in momentum space is

$$[(p^2 - m^2)\eta^{\mu\nu} + 2(k_F)^{\mu\alpha\nu\beta}p_\alpha p_\beta] \epsilon_\nu = 0, \quad (2)$$

where ϵ^μ is the polarization vector. One implication of this equation is found by dotting with p_μ , yielding the condition

$$(p^2 - m^2)(\epsilon \cdot p) = 0, \quad (3)$$

indicating that states that satisfy a perturbed dispersion relation must be transverse and any state with $\epsilon \cdot p \neq 0$ must satisfy the conventional Lorentz covariant dispersion relation. Setting the determinant of the coefficient matrix of (2) to zero yields an eighth-order

polynomial in p^0 that must be solved to yield the energies. This dispersion relation can be expressed in terms of scalar invariant powers of the perturbation $K^{\mu\nu} = 2k_F^{\mu\alpha\nu\beta} p_\alpha p_\beta$. For example, when $K^\mu{}_\mu = 0$ (this is the case for k_F anti-self-dual and therefore birefringent, a fact to be explained later in the paper), the dispersion relation takes the relatively simple form

$$[p^2 - m^2] [6(p^2 - m^2)^3 - 3(p^2 - m^2)\text{Tr } K^2 + 2\text{Tr } K^3] = 0, \quad (4)$$

where $\text{Tr } K^2 = K^\mu{}_\nu K^\nu{}_\mu$ and $\text{Tr } K^3 = K^\mu{}_\nu K^\nu{}_\alpha K^\alpha{}_\mu$. Note that $K^{\mu\nu} p_\nu = 0$ indicating that the fourth-order term must vanish as the null-space of K is always at least one dimensional, implying that its determinant is identically zero. A symmetry under $p^\mu \rightarrow -p^\mu$ reduces the number of independent solutions to four after reinterpretation of the negative energy solutions as positive energy ones. We also note that a factor of p^2 can be extracted from the $\text{Tr } K^3$ term using an argument discussed in [11] involving the dual action of $\Lambda^3 K$ on the momentum vector. This means that in the massless limit, a factor of p^4 can be factored out of the dispersion relation, in agreement with the results of [11]. This equation (for $m \neq 0$) normally has four independent solutions which are labeled using $\lambda = 0, 1, 2, 3$. An orthogonality relation can be derived using the equation of motion in the form

$$[\omega_p^2 \eta^{\mu\nu} + 2k_F^{\mu 0 i \nu} p_i p_j] \epsilon_\nu^{(\lambda)}(\vec{p}) = \left[(p_0^{(\lambda)})^2 \tilde{\eta}^{\mu\nu} - 2\hat{k}_F^{\mu 0 i \nu} p_0^{(\lambda)} p_i \right] \epsilon_\nu^{(\lambda)}(\vec{p}), \quad (5)$$

with $\omega_p = \sqrt{p^2 + m^2}$ and $\hat{k}_F^{\mu 0 i \nu} = k_F^{\mu 0 i \nu} + k_F^{\nu 0 i \mu}$. Note that all of the dependency on p_0 has been moved to the right-hand side of the equation. This equation can then be multiplied by $\epsilon_\mu^{(\lambda')}(\vec{p})$ on the left and subtracted from the corresponding equation with $\lambda \leftrightarrow \lambda'$ switched. The result is the following orthogonality relation which is valid when $p_0^{(\lambda)} \neq p_0^{(\lambda')}$

$$\epsilon_\mu^{(\lambda')}(\vec{p}) \left[\left(p_0^{(\lambda)} + p_0^{(\lambda')} \right) \tilde{\eta}^{\mu\nu} - 2 \left(k_F^{\mu 0 i \nu} + k_F^{\nu 0 i \mu} \right) p_i \right] \epsilon_\nu^{(\lambda)}(\vec{p}) = 2p_0^{(\lambda)} \eta^{\lambda\lambda'}, \quad (6)$$

and the normalization has been chosen to match the conventional case with $\lambda = 0$ labeling the timelike vector. A similar orthogonality relation exists between polarization vectors with opposite three-momenta

$$\epsilon_\mu^{(\lambda')}(-\vec{p}) \left[\left(p_0^{(\lambda')}(-\vec{p}) + p_0^{(\lambda)}(\vec{p}) \right) \tilde{\eta}^{\mu\nu} + 2 \left(k_F^{\mu 0 i \nu} + k_F^{\nu 0 i \mu} \right) p_i \right] \epsilon_\nu^{(\lambda)}(\vec{p}) = 0. \quad (7)$$

Note that for parity-violating coefficients (k_F^{0ijk}) one has generally $p_0^{(\lambda)}(\vec{p}) \neq p_0^{(\lambda)}(-\vec{p})$ in contrast to the conventional case, so care must be taken to distinguish the various quantities.

QUANTIZATION

The canonical momentum is computed in the usual way by taking derivatives of Lagrangian (1) with respect to the \dot{A}_μ fields

$$\Pi^j = F^{j0} + k_F^{j0\alpha\beta} F_{\alpha\beta}, \quad \Pi^0 = -\partial_\mu A^\mu. \quad (8)$$

Imposing equal-time canonical commutation rules

$$[A_\mu(t, \vec{x}), \Pi^\nu(t, \vec{y})] = i g_\mu{}^\nu \delta^3(\vec{x} - \vec{y}), \quad (9)$$

along with

$$[A_\mu(t, \vec{x}), A_\nu(t, \vec{y})] = [\Pi^\mu(t, \vec{x}), \Pi^\nu(t, \vec{y})] = 0, \quad (10)$$

implements the standard canonical quantization in a covariant manner as is done in the conventional Gupta-Bleuler method. This implies that the time derivatives of the spatial components A^i satisfy the modified commutation relations

$$[\dot{A}^i(t, \vec{x}), A^j(t, \vec{y})] = -i R^{ij} \delta^3(\vec{x} - \vec{y}) \quad (11)$$

where R^{ij} is the inverse matrix of $\delta^{ij} - 2(k_F)^{0i0j}$. In any concordant frame where k_F is reasonably small, this inverse exists. The commutation relations involving \dot{A}^0 and A^i are the same as in the usual case, so for future use, it is convenient to define a covariant-looking tensor $\bar{\eta}^{\mu\nu}$ by setting $\bar{\eta}^{00} = 1$, $\bar{\eta}^{0i} = 0$, and $\bar{\eta}^{ij} = -R^{ij}$. The commutation relations take the form

$$[\dot{A}^\mu(t, \vec{x}), A^\nu(t, \vec{y})] = i \bar{\eta}^{\mu\nu} \delta^3(\vec{x} - \vec{y}). \quad (12)$$

This matrix is also the inverse of $\tilde{\eta}^{\mu\nu} = \eta^{\mu\nu} - 2k_F^{\mu 0 0 \nu}$ as $\bar{\eta}^{\mu\nu} \tilde{\eta}_{\nu\alpha} = \eta^\mu{}_\alpha$. Note also that the time derivatives of A no longer commute, rather

$$[\dot{A}^\mu(t, \vec{x}), \dot{A}^\nu(t, \vec{y})] = -2i \bar{\eta}_{\mu\alpha} (k_F^{\alpha 0 i \beta} + k_F^{\beta 0 i \alpha}) \bar{\eta}_{\beta\nu} \frac{\partial}{\partial x^i} \delta^3(\vec{x} - \vec{y}), \quad (13)$$

involving the spatial derivatives of the delta function.

The field can then be expanded in terms of Fourier modes as

$$A_\mu(x) = \int \frac{d^3 \vec{p}}{(2\pi)^3} \sum_\lambda \frac{1}{2p_0^{(\lambda)}} (a^\lambda(\vec{p}) \epsilon_\mu^{(\lambda)}(\vec{p}) e^{-ip \cdot x} + a^{\lambda\dagger}(\vec{p}) \epsilon_\mu^{(\lambda)}(\vec{p}) e^{ip \cdot x}). \quad (14)$$

This formula may be inverted using the orthogonality relations (6) and (7) as

$$a^\lambda(\vec{q}) = \eta^{\lambda\lambda'} \int d^3 \vec{x} e^{iq \cdot x} \left[i \overleftrightarrow{\partial}_0 \tilde{\eta}^{\mu\nu} - 2(k_F^{\mu 0 i \nu} + k_F^{\nu 0 i \mu}) q_i \right] \epsilon_\nu^{(\lambda')}(\vec{q}) A_\mu(x), \quad (15)$$

and

$$a^{\lambda\dagger}(\vec{q}) = \eta^{\lambda\lambda'} \int d^3\vec{x} e^{-iq\cdot x} \left[-i\overset{\leftrightarrow}{\partial}_0 \tilde{\eta}^{\mu\nu} - 2(k_F^{\mu 0 i\nu} + k_F^{\nu 0 i\mu}) q_i \right] \epsilon_\nu^{(\lambda')}(\vec{q}) A_\mu(x). \quad (16)$$

These can be used to compute

$$[a^\lambda(\vec{p}), a^{\lambda'\dagger}(\vec{q})] = -(2\pi)^3 2p_0^{(\lambda)} \eta^{\lambda\lambda'} \delta^3(\vec{p} - \vec{q}), \quad (17)$$

as well as

$$[a^\lambda(\vec{p}), a^{\lambda'}(\vec{q})] = [a^{\lambda\dagger}(\vec{p}), a^{\lambda'\dagger}(\vec{q})] = 0, \quad (18)$$

demonstrating that the raising and lowering operators obey conventional bosonic statistical relations.

It remains to impose the restriction on the physical states that eliminates the negative norm states as in the conventional Gupta-Bleuler approach. As is shown in the appendix, the physical states still satisfy the usual condition

$$\langle \psi_{\text{phys}} | \partial_\mu A^\mu | \psi_{\text{phys}} \rangle = 0. \quad (19)$$

Application of the basic commutators (17) and (18) to the coordinate-space commutation relations (10), (12), and (13) for the vector potential imply a variety of modified completeness relations for the polarization vectors, including

$$\frac{1}{2} \sum_{\lambda, \lambda'} \eta_{\lambda\lambda'} \left[\epsilon_\mu^{(\lambda)}(\vec{p}) \epsilon_\nu^{(\lambda')}(\vec{p}) + \epsilon_\mu^{(\lambda)}(-\vec{p}) \epsilon_\nu^{(\lambda')}(-\vec{p}) \right] = \bar{\eta}_{\mu\nu}. \quad (20)$$

Note that the right-hand side is independent of momentum indicating that a complete set of polarization vectors must exist for every momentum choice. An explicit example is presented later in the paper for which the above completeness relation holds. Unfortunately, this is not always true in the massless limit as will be seen in the explicit example, so the presence of a mass term is crucial for the consistency of the quantization procedure.

THE PROPAGATOR

In Lorentz-invariant field theory the vacuum expectation value of the time-ordered product of two field operators plays a central role as it is a Green's function (the Feynman propagator). It is important to check that this is still the case for the time-ordered product

$\langle 0|TA_\mu(x)A_\rho(y)|0\rangle$. Indeed, by acting with the kinetic operator one obtains

$$\begin{aligned}
& ((\partial^2 + m^2)\eta^{\mu\beta} - 2k_F^{\mu\nu\alpha\beta}\partial_\nu\partial_\alpha)\langle 0|A_\beta(x)A_\rho(y)\theta(x^0 - y^0) + A_\rho(y)A_\beta(x)\theta(y^0 - x^0)|0\rangle = \\
& = \delta(x^0 - y^0)(\eta^{\mu\beta}\eta^{0\alpha} - k_F^{\mu 0\alpha\beta} - k_F^{\mu\alpha 0\beta})\langle 0|[\partial_\alpha A_\beta(x), A_\rho(y)]|0\rangle \\
& = \delta(x^0 - y^0)\tilde{\eta}^{\mu\beta}\langle 0|[\dot{A}_\beta(x), A_\rho(y)]|0\rangle \\
& = \delta(x^0 - y^0)\tilde{\eta}^{\mu\beta}i\tilde{\eta}_{\beta\rho}\delta^3(\vec{x} - \vec{y}) \\
& = i\delta^4(x - y)\delta_\rho^\mu,
\end{aligned} \tag{21}$$

verifying that in fact the time-ordered product still functions as the appropriate Green's function for the perturbed equation of motion. In the first identity we used the fact that $A_\mu(x)$ satisfies the equation of motion, so that the only nonzero contributions arise when at least one time-derivative acts on the delta-functions.

Using the mode expansion (14), it follows that

$$\begin{aligned}
\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = & - \int \frac{d^3p}{(2\pi)^3} \sum_{\lambda,\lambda'} \frac{\eta_{\lambda\lambda'}}{2p_0^{(\lambda)}(\vec{p})} \epsilon_\mu^{(\lambda)}(\vec{p}) \epsilon_\nu^{(\lambda')}(\vec{p}) \\
& \times (e^{-ip\cdot(x-y)}\theta(x^0 - y^0) + e^{ip\cdot(x-y)}\theta(y^0 - x^0)),
\end{aligned} \tag{22}$$

expressing the propagator explicitly in terms of a polarization sum. Eq. (22) complements the covariant expressions obtained in [8] for the propagator.

HAMILTONIAN FORMALISM

Calculation of the hamiltonian density $\mathcal{H} = \Pi^\mu \dot{A}_\mu - \mathcal{L}$ yields (after appropriate partial integrations with $H = \int d^3\vec{x}\mathcal{H}$),

$$\begin{aligned}
\mathcal{H} = & \frac{1}{2} \left((\partial_j A^j)^2 - (\dot{A}^0)^2 + A^0 \partial_0 (\partial \cdot A) + \vec{E}^2 + \vec{B}^2 \right) \\
& - k_{0i0j} F^{0i} F^{0j} + \frac{1}{4} k_{ijkl} F^{ij} F^{kl} - \frac{1}{2} m^2 A_\mu A^\mu.
\end{aligned} \tag{23}$$

The commutation relations fix the action of the hamiltonian on fields to be the conventional relation

$$i[H, A_\mu] = \partial_0 A_\mu, \tag{24}$$

ensuring that the appropriate zero components of momentum are the energy eigenvalues. The momentum operator can be constructed using the conserved energy-momentum tensor

as

$$P^i = \int d^3\vec{x} \left(\Pi^j \partial^i A_j - (\partial \cdot A) \partial^i A^0 \right), \quad (25)$$

and satisfies

$$i[P^i, A_\mu] = \partial^i A_\mu. \quad (26)$$

CLASSIFICATION OF k_F TERMS

A useful classification of the k_F tensors involves a splitting of the tensor into self-dual and anti-self dual pieces.

$$k_F = k_F^{SD} \oplus k_F^{ASD}, \quad (27)$$

where the dual of k_F is defined by

$$(\tilde{k}_F)^{\mu\nu\alpha\beta} = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} (k_F)_{\rho\sigma\gamma\delta}. \quad (28)$$

A straightforward calculation reveals that the self-dual condition $k_F = \tilde{k}_F$ corresponds exactly to the nonzero single trace components of k_F , while the anti-self dual condition corresponds to the trace-free condition on k_F . One other useful fact is that the F^2 term in the lagrangian corresponds to a choice of an anti-self dual k -term of the form

$$\eta^{\mu\nu\alpha\beta} \equiv \frac{1}{2} (\eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\mu\beta} \eta^{\nu\alpha}), \quad (29)$$

which is typically set to zero to avoid a Lorentz-preserving correction to the kinetic term of the lagrangian. The trace components can be most easily handled using coordinate redefinition techniques [12], at least when dealing with the free theory. Application of the anti-self dual condition on k_F implies that the $(k_F)^{ijkl}$ coefficients can be expressed in terms of the $(k_F)^{0i0j}$. In addition, the only nonvanishing $(k_F)^{0ijk}$ components are the ones with $i \neq j \neq k$. This decomposition has previously been pointed out in [13] where the trace terms were anti-self dual, opposite the current application, due to the fact that Euclidean space was used in the instanton theory rather than Minkowski space.

ILLUSTRATIVE SINGLE-PARAMETER EXAMPLE

The above formalism is well-illustrated by the following exactly solvable example involving only one non-vanishing parameter. The fundamental impediment to covariant quantization

of the massless photon can also easily be seen in this example. The $k_F^{\mu\nu\alpha\beta}$ tensor is selected so that $k_F^{0101} = k/2$, and $k_F^{2323} = -k/2$ to satisfy the anti-self-dual condition. The natural symmetries of k_F are also imposed yielding $k_F^{1001} = -k_F^{0101} = \dots$. The double trace term $(k_F)^{\mu\nu}_{\mu\nu}$ is not zero for this choice, but it can be made zero by adding an appropriate overall trace term to k_F

$$(k'_F)^{\mu\nu\alpha\beta} = (k_F)^{\mu\nu\alpha\beta} + \frac{k}{3}\eta^{\mu\nu\alpha\beta}, \quad (30)$$

where $\eta^{\mu\nu\alpha\beta} = (1/2)(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\eta^{\nu\alpha})$.

The dispersion relation factors nicely into four separate terms

$$(p^2 - m^2)(p_0^2 - \omega_1^2)(p_0^2 - \omega_2^2)(p_0^2 - \omega_3^2) = 0, \quad (31)$$

where $\omega_1^2 = \bar{p}^2 + m^2/(1 + k/3)$, $\omega_2^2 = \bar{p}^2 + (m^2 + k(p_2^2 + p_3^2))/(1 - 2k/3)$, and $\omega_3^2 = \bar{p}^2 + (m^2 - k(p_2^2 + p_3^2))/(1 + k/3)$, are perturbed versions of $\omega_p^2 = \bar{p}^2 + m^2$. The polarization vectors associated with each energy are,

$$\epsilon_\mu^{(0)} = \frac{1}{m} \begin{pmatrix} \omega_p \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}, \quad \epsilon_\mu^{(1)} = \frac{1}{m\sqrt{(\omega_1^2 - p_1^2)(p_2^2 + p_3^2)}} \begin{pmatrix} \omega_1(p_2^2 + p_3^2) \\ p_1(p_2^2 + p_3^2) \\ p_2(\omega_1^2 - p_1^2) \\ p_3(\omega_1^2 - p_1^2) \end{pmatrix}, \quad (32)$$

$$\epsilon_\mu^{(2)} = \frac{1}{\sqrt{(1 - 2k/3)(\omega_2^2 - p_1^2)}} \begin{pmatrix} p_1 \\ \omega_2 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon_\mu^{(3)} = \frac{1}{\sqrt{(1 + k/3)(p_2^2 + p_3^2)}} \begin{pmatrix} 0 \\ 0 \\ -p_3 \\ p_2 \end{pmatrix}, \quad (33)$$

where the normalizations are chosen so that Eq. (20) is satisfied. When the mass is nonzero, there are always one timelike and three spacelike polarization vectors. When the mass is identically zero, an issue can be seen by examining the polarization vectors $\epsilon^{(0)}$ and $\epsilon^{(1)}$. The normalizations can be adjusted so that there are no divergences in this limit and the two polarization vectors become degenerate, both being proportional to the same vector. There is in fact no state that satisfies $\epsilon \cdot p \neq 0$, a curious implication of the existence of a nontrivial Lorentz-violation parameter. A direct consequence of this is that $\Pi^0 = -\partial \cdot A$ vanishes identically, thus obstructing covariant quantization. This example illustrates how the mass regulator alleviates this degeneracy issue and ensures that the covariant quantization procedure is consistent.

SUMMARY

Direct application of the Gupta-Bleuler covariant quantization procedure to SME photon sector runs into fundamental issues. Specifically, the conjugate momentum Π^0 introduced by adding the gauge-fixing term is generally forced to be zero in the presence of the perturbation term, making the covariant quantization procedure fail. On the other hand, covariant quantization of the photon field appears to be consistent when a mass term is included in the Lagrangian. This is certainly the case when the mass term dominates the Lorentz-violating terms $m \gg \sqrt{k_F} p_0$. In fact, in all of the specific examples that we looked at, any nonzero mass term actually ensured the existence of one time-like and three space-like polarization vectors for arbitrary three-momentum. We hypothesize that this is the case for arbitrary perturbative k_F choices. As future work, it would be of significant interest to confirm or deny this hypothesis. If true, then it implies that a consistent method for performing quantum-field-theoretic calculations in the SME photon sector can be implemented by including a mass regulator term and taking the appropriate zero mass limit at the end of the calculation. This procedure is already commonly applied to regulate infrared divergences in QED.

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APPENDIX: BRST AND THE CONDITION ON THE PHYSICAL STATES

The structure of the space of states is most clearly established by completing the photon lagrangian (1) by adding the contributions from a Stueckelberg scalar field ϕ as well as the (anticommuting) ghost and antighost fields c and \bar{c} [8]:

$$\mathcal{L}_{Stueck} = \mathcal{L}_A + \mathcal{L}_\phi + \mathcal{L}_{gh} \quad (34)$$

where \mathcal{L}_A is given by (1) and

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\xi m^2 \phi^2 \quad (35)$$

and

$$\mathcal{L}_{gh} = -\bar{c}(\partial^2 + \xi m^2)c. \quad (36)$$

Lagrangian (34) can now be obtained from the lagrangian

$$\mathcal{L}'_{Stueck} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}k_F^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} + \frac{1}{2}m^2(A_\mu - \frac{1}{m}\partial_\mu\phi)^2 + \frac{\xi}{2}B^2 + B(\partial_\mu A^\mu + \xi m\phi) - \bar{c}(\partial^2 + \xi m^2)c \quad (37)$$

upon integrating out the (auxiliary) Nakanishi-Lautrup field B , introduced for consistency of the BRST transformation.

Lagrangian (37) is invariant under the BRST transformation s defined by

$$sA_\mu = \epsilon\partial_\mu c \quad (38)$$

$$s\phi = \epsilon mc \quad (39)$$

$$sc = 0 \quad (40)$$

$$s\bar{c} = \epsilon B \quad (41)$$

$$sB = 0 \quad (42)$$

where ϵ is some infinitesimal grassman-valued gauge parameter. Note that s is nilpotent off-shell: $s^2 = 0$. The general structure is in fact unaltered by the presence of the Lorentz-violating terms, mainly since gauge invariance is preserved by the k_F term.

As is customary, we now define the set of *exact states* \mathcal{H}_2 as the states that can be obtained as the image of s acting on some other state. If we substitute the field B through its equation of motion by $-\xi^{-1}(\partial_\mu A^\mu + \xi m\phi)$ we see that \mathcal{H}_2 consists of the states built of the vacuum by acting with the field oscillators either of the ghost c or of the field combination $\partial_\mu A^\mu + \xi m\phi$. The set of *physical states* \mathcal{H}_0 is defined as the set of (closed) states annihilated by the BRST operator s modulo \mathcal{H}_2 . Note that this procedure makes sense because any exact state, while closed (because $s^2 = 0$), is automatically orthogonal to any closed state: $\langle exact | closed \rangle = 0$. The remaining set \mathcal{H}_1 of *unphysical states* are those that are not closed. It consists of states built out of the oscillators of the antighost, the longitudinal mode of A_μ ($\lambda = 0$) and the scalar ϕ , with the exception of the combination $\partial_\mu A^\mu + \xi m\phi$. Thus we conclude that the exact mode is a linear combination of the longitudinal mode and the scalar ϕ , while the linear combination orthogonal to this is unphysical. The (remaining) physical states are the three transverse modes of A_μ .

The only mode for which $p^\mu \epsilon_\mu^{(\lambda)}(\vec{p})$ does not vanish is the longitudinal mode $\lambda = 0$. This fact follows from examination of the equation of motion in an observer frame for which $\vec{p} = 0$ which generally exists when a mass term is present, at least for the case of time-like momenta. In this frame, the equation of motion for ϵ_μ reduces to

$$(p_0^2 \tilde{\eta}^{\mu\nu} - m^2 \eta^{\mu\nu}) \epsilon_\nu = 0, \quad (43)$$

where $\tilde{\eta}^{0i} = \tilde{\eta}^{i0} = 0$ and $\tilde{\eta}^{00} = 1$. It is clear by inspection that there is one time-like polarization satisfying $p_0^2 = m^2$ and three space-like polarizations satisfying $p_0^2 = m^2/(1 - 2k_{\Lambda F}^{0i0i})$, $i = \{1, 2, 3\}$ with no sum, where the spatial orientation of the coordinate system has been rotated to diagonalize $\tilde{\eta}^{ij}$ in the boosted frame. The parameter $k_{\Lambda F}$ represents the boosted value of the corresponding lorentz-violating parameters in this special frame. The structure of the solution space is clearly seen in this frame where the time-like polarization vector points along the momentum and the polarizations corresponding to the perturbed dispersion relations point in space-like directions and satisfy the transversality condition $\epsilon \cdot p = 0$. An observer Lorentz transformation can then be used to boost back to the original frame maintaining the properties that $\epsilon_\mu^{(0)}$ is proportional to p_μ and the other states are orthogonal to p_μ .

An important caveat to the above argument is that the boost cannot get too large or the $k_{\Lambda F}$ parameters may grow to order one leading to a breakdown of the perturbative theory (also called boosting to a nonconcordant frame [14]). In the present context, this can occur when the photon energy $p_0 \sim m/\sqrt{k_F}$, where k_F is of order of the elements of the lorentz-violating couplings.

In the massless case, the longitudinal mode $\lambda = 0$ is no longer time-like, but light-like, as $p^2 = m^2 = 0$ for this state. Any exact mode ($\partial_\mu A^\mu \neq 0$) will then no longer involve the longitudinal mode, but possibly one of the other modes, which must necessarily satisfy light-like dispersion relation $p^2 = 0$, due to (3) (because otherwise we would have $\epsilon \cdot p = 0$). In fact, generically this mode does not even exist as is shown in the special example presented in the body of the paper. The problem arises when the equation of motion is applied with $p^2 = 0$ resulting in the condition $K^{\mu\nu} \epsilon_\nu = 0$. It is easy to demonstrate that $K^{\mu\nu}$ is generically of rank 3 (except for certain special choices of \vec{p}) which implies that the only polarization with $p^2 = 0$ is the gauge state with polarization proportional to the momentum. This leads to the conclusion that there is no exact state in these cases.

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